Analyzing Mathematical Knowledge using Network Theory: Implications for Mathematics Education

ELIZABETH MOWAT
University of Alberta, Canada

Abstract

As a new approach to analyzing complex systems, network theory may provide a useful frame for informing mathematics education. Using cognitive mechanisms outlined in the theory of embodied mathematics, due to Lakoff and Núñez, to offer an interpretation of the nature and association of mathematical concepts, a possible organization of the structure of mathematics is explored. This metaphoric network displays the scale-free topology and consequent network dynamics characteristic of many complex systems. It may be both instructive and important to use the network structure of mathematical knowledge to shed light both on cognition in mathematics and on mathematics education; implications for classroom teaching and curriculum are discussed.

Introduction

Throughout the history of modern schooling, mathematics education has been organized around prevailing beliefs as to the nature of mathematics. For many centuries, the philosophies of Plato, Euclid and Descartes—viewing mathematics as composed of rigid, orderly, and hierarchical propositions—influenced the structure of mathematics curricula. More recently, the work of the Formalists in the early twentieth century was taken up within the New Math curriculum as a focus on axioms, laws and proofs.

As a new development in mathematics, arising in the last decade, the field of network theory may present a novel way of understanding the structure of mathematical knowledge and, in consequence, of informing pedagogy. Briefly, network

Proceedings of the 2007 Complexity Science and Educational Research Conference
Feb 18–20 • Vancouver, British Columbia • pp. 165–176 • www.complexityandeducation.ca

theory examines the various ways in which a group of objects can be connected in some fashion. Originally developed as a branch of applied mathematics, techniques developed in network theory have been used to explore complex systems in nature, society and the human body. The conclusions of such network analyses arise from investigating the underlying topology of complex systems, rather than from simply examining the particular entities that they comprise (Barabási, 2003).

Viewing the elements of a system as vertices or *nodes* in a network and their interactions as *links* among nodes, the elements of a complex system and their connections can be represented by a graph. Using this technique, Watts and Strogatz (1998) and Barabási and Albert (1999) found patterns previously unnoticed in complex phenomena and formulated simple, yet comprehensive, laws that describe network structure and dynamics. Although the use of network theory in analyzing complex systems is rapidly expanding, it has not yet been applied to the field of mathematics education. It would seem desirable that educators be aware of these powerful methods of analysis that have proved invaluable in explaining how and why complex systems behave as they do.

I suggest that a possible network structure for mathematical knowledge may be found in the *theory of embodied mathematics* as put forth specifically by Lakoff and Núñez (2000). They describe mathematics as being determined by interactions between the human body, the brain and the environment, as involving largely unconscious cognitive mechanisms, and as metaphoric in nature. It is extended from a rather limited set of inborn skills to an ever-growing collection of *conceptual domains* or coherent organizations of experience. These are connected by *conceptual metaphors* that project patterns of inference from one domain to another. Lakoff and Núñez (2000) assert that these cognitive mechanisms play a key role in providing the basis for mathematical knowledge. I propose that considering conceptual domains as the nodes of a network and conceptual metaphors as the links between these nodes may offer an appropriate and productive model for analysis of the structure of mathematics. In this paper, I will refer to this model as the *metaphoric network of mathematics*.

Network Structure

The structure of a network is influenced by the nature of its nodes and the links that connect them. In the metaphoric network of mathematics, the nodes or conceptual domains are formed from physical and mental experiences (Lakoff & Núñez, 2000). Thus, even the simplest of conceptual domains possesses considerable internal structure (Johnson, 1987). Each domain contains interconnected elements forming a complex of sensory perceptions, language and related concepts, which are all held together by a central coordinating conceptual node (Lamb, 1999). For example, the conceptual domain of CIRCLE[1] (see Figure 1) contains many nodes representing the different aspects of a person's knowledge of circles. When enough nodes in the domain are activated, they operate together in gestalt and form an individual's perception of the circle (Lakoff, 1987; Lakoff & Núñez, 2000; Lamb, 1999). Any depiction of a conceptual domain must necessarily be incomplete, for any concept may contain thousands of nodes (Lamb, 1999) and continually changes with new experiences.

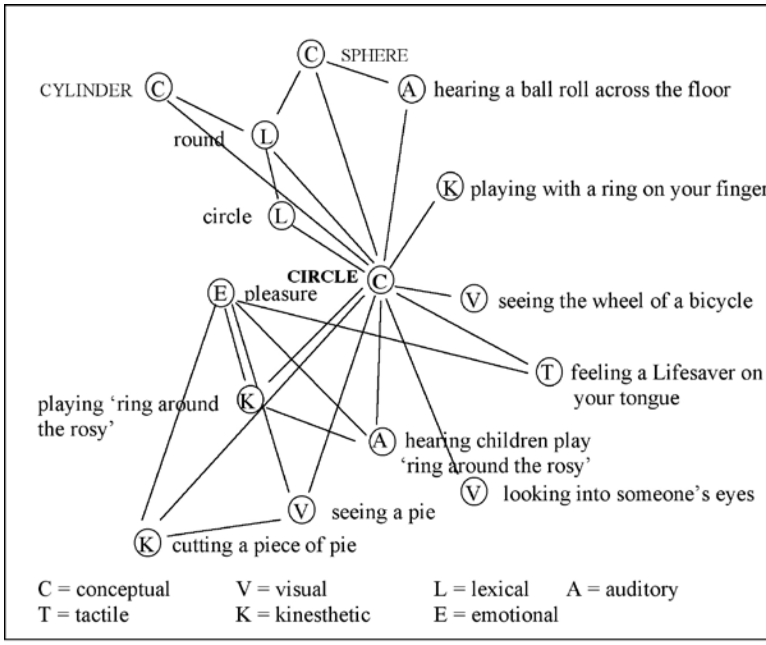

FIGURE 1. Part of the conceptual domain of CIRCLE (Mowat & Davis, in submission)

Viewed from this perspective, nodes in the metaphoric web of mathematics are not just basic units, but can be understood as subnetworks of the larger structure (Kimmel, 2002; Kövacses, 2002; Lamb, 1999). Thus, the structure of mathematics reveals the characteristic pattern of nested forms (see Figure 2) that typifies complex systems (Davis & Simmt, 2006; Davis & Sumara, 2000).

Links in the metaphoric network of mathematics are conceptual metaphors. These cognitive mechanisms project inferential structure from one conceptual domain (the *source*) to another (the *target*), enabling people to reason about new and abstract concepts in terms of more familiar and concrete ideas (Lakoff & Núñez, 2000). To illustrate, the SET IS A CONTAINER[2] metaphor (see Figure 3) projects the

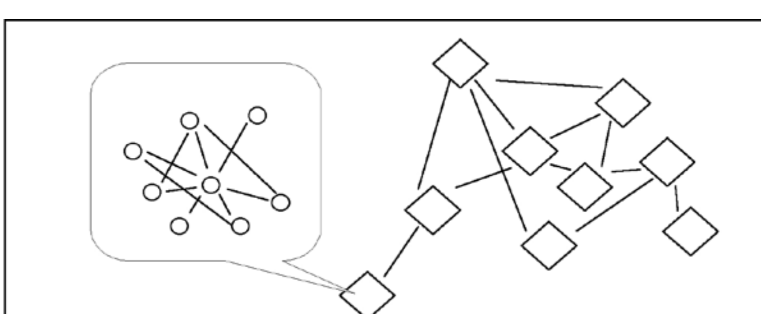

FIGURE 2. A subnetwork representing a conceptual domain (like CIRCLE), which is nested within the metaphoric network of mathematics

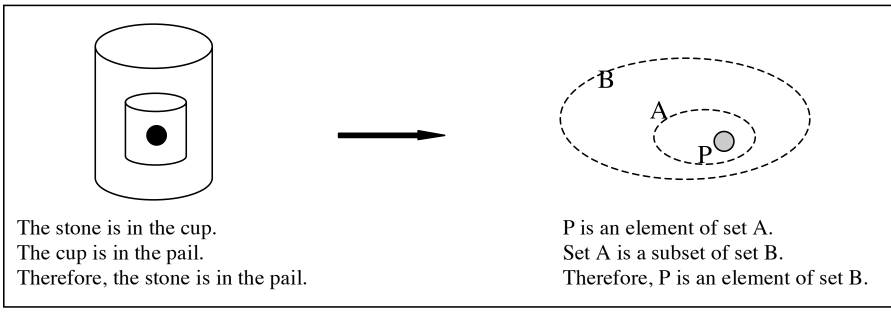

FIGURE 3. Reasoning from physical experience transferred to abstract mathematics through the SET IS A CONTAINER conceptual metaphor (Mowat, 2005).

logic inherent in physical containers onto 'cognitive' containers or sets. Modes of thought developed through bodily experiences involving material containers can thus be used to reason about nonspatial situations (Johnson, 1987).

Patterns of connections in this web of metaphors and concepts are not simple. For many conceptual domains, a single metaphor is unable to provide enough structure to depict the concept as a whole (Lakoff & Johnson, 1999). More than one source is required to illuminate such rich domains, each metaphor describing a different aspect of the complex target (Kimmel, 2002; Kövacses, 2002). For example, Lakoff & Núñez (2000) describe how ARITHMETIC is structured through "forming collections, putting objects together, using measuring sticks, and moving through space" (p. 102). The four grounding metaphors that result—ARITHMETIC IS OBJECT COLLECTION, ARITHMETIC IS OBJECT CONSTRUCTION, THE MEASURING STICK METAPHOR, and ARITHMETIC IS MOTION ALONG A PATH—enable the extension of rather limited innate numerical abilities into basic arithmetic.

Just as target domains may be metaphorically linked to more than one source, many source domains provide a framework for a variety of targets (Kövacses, 2002). Key features of the source are transferred to each of the target domains. Consider the wide-ranging influence of a concept like SET, which is connected to many mathematical ideas (some of which are listed in Figure 4). The axioms of set theory provide these domains with a unifying structure and provide a foundation for modern mathematics.

Although often considered to be directional mappings from source to target (Lakoff & Núñez, 2000), metaphors do not always project inferential structure in just one direction (Danziger, 1990; Gentner, Bowdle, Wolff & Boronat, 2001; Meisner, 1992). Through the interactions that continually take place between the two domains, the target may gradually change understanding of the original source

AN ORDERED PAIR IS A SET.	A NUMBER IS A SET.
A FUNCTION IS A SET.	A LINE IS A SET.
A LOGICAL PROPOSITION IS A SET.	A GRAPH IS A SET.

FIGURE 4. Metaphors with a source domain of SET

domain that shaped it, causing the conceptual metaphor to become bidirectional (Kimmel, 2002; English, 1997; Sfard, 1997).

Network Topology

It is evident that the nodes and links of the metaphoric network of mathematics form an intricate web. As a complex system, this structure is likely to display a *scale-free* topology (Barabási, 2003; Watts, 2002). For graphs with this pattern (see Figure 5), it is not possible to determine the *scale,* that is, the typical number of links to a node. While a few nodes are very highly connected, most are linked to only a few other vertices.

The nodes that possess the greatest number of links are the *hubs* of a scale-free network. Clusters are formed within which every vertex is connected to a hub; these key nodes are in turn linked to more central nodes, and so on. Portrayals of scale-free networks representing real-world systems tend to be much more complex than the structure shown in Figure 5 (see http://www-personal.umich.edu/~mejn/networks/ for examples).

In order for a network to exhibit this type of pattern, it must possess certain characteristics. Barabási and Albert (1999) state that *growth* and *preferential attachment* are necessary and sufficient conditions for a network to be scale-free. I suggest that the metaphoric network of mathematics does possess these attributes.

It is clear that the system of mathematics grows; new concepts are continually being added to its structure. The nature of conceptual metaphors can cause this increase in several ways. Metaphors have the power to create new concepts (Boyd, 1979; Chiu, 2000; Presmeg, 1997; Sfard, 1997). The four grounding metaphors described by Lakoff and Núñez (2000) are classic examples of this type of constitutive metaphor in mathematics.

Extensions of established metaphors may lead to new understandings. For example, the MEASURING STICK METAPHOR portrays natural numbers as physical segments that can be placed end to end to measure objects (Lakoff & Núñez, 2000).

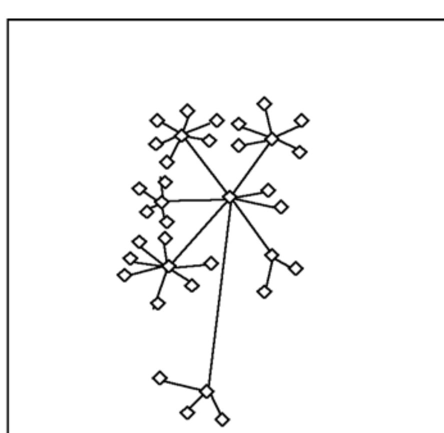

FIGURE 5. A much-simplified network displaying a scale-free topology

FIGURE 6. Grounding √2 and π using the MEASURING STICK METAPHOR

Stretching this metaphor a bit farther, any line that can be measured in any way, using any device, can be considered a number. This provides a metaphoric foundation for a previously unknown domain, the irrational numbers (see Figure 6).

Nodes are also added to the network through conceptual blends. Conceptual blends construct a partial correspondence between two unrelated sources and project this correlation onto a new domain (Fauconnier & Turner, 1998; Lakoff & Núñez, 2000). The blended concept is more than merely a combination of the original sources. To illustrate, the UNIT CIRCLE is the conceptual blend of a circle in the Euclidean plane and the Cartesian plane with coordinate axes (see Figure 7).[3] It possesses characteristics of both of these domains, but also has emergent properties related to trigonometry that are not found in either of the contributing domains.

The property of preferential attachment is also found in the metaphoric network of mathematics; a disproportionately large number of links tend to be made to nodes that are already highly connected. Several factors have been shown to determine a concept's 'attractiveness' for new connections. The 'age' of a node plays a role; ideas added to the network early in its development have more time to acquire links (Barabási, 2003; Barabási, Albert, Jeong & Bianconi, 2000). Other concepts may exhibit

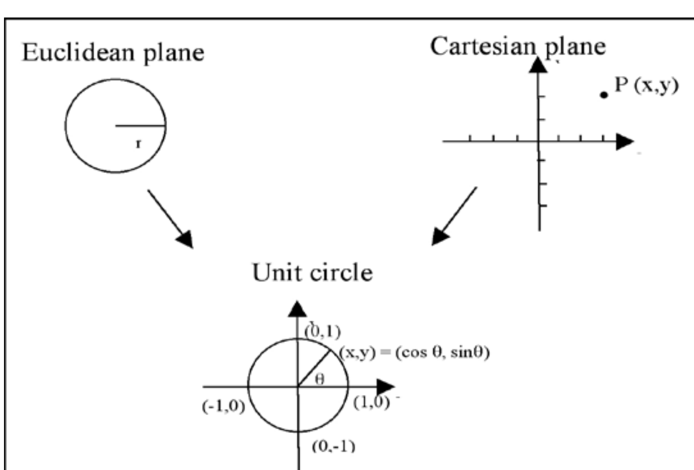

FIGURE 7. Features of Euclidean and Cartesian geometry combined into the UNIT CIRCLE. (Mowat, 2005)

a greater degree of 'fitness' (Bianconi & Barabási, 2001). Their inferential structure, like that of the concept of SET, may be powerful enough to provide useful structure for many different domains. Familiarity also plays a role; people tend to use ideas with which they are already acquainted (Barabási, 2003). Whatever the cause, source domains with numerous links are likely to attract even more connections.

Network Dynamics and Mathematical Understanding

By demonstrating growth and preferential attachment, the metaphoric network of mathematics is shown to possess a scale-free structure, the common pattern that governs the attributes and dynamics of real-world complex systems (Barabási, 2003; Watts, 2002). Such networks are generally very robust. Since the majority of nodes have only a few links, a significant number can be removed from the system with little or no damage (Barabási, 2003; Watts, 2002). If a hub is compromised, however, the effect is much greater (see Figure 8). Nodes directly connected to the hub fail first, nodes linked to these fall next and so on, like an extended pattern of dominos. While this process, called a *cascading failure*, can go unnoticed for a long time, the collapse of one highly connected node eventually causes the network to break into fragments (Albert, Jeong, & Barabási, 2000; Barabási, 2003).

The susceptibility of scale-free networks to cascading failures may offer an explanation for some of the difficulties experienced by students in learning mathematics. The metaphoric network of mathematics is inherently vulnerable because of the crucial role that certain concepts play in ensuring the connectivity of the system. If one of these hubs fails, the possibility exists that a student's comprehension of certain connected conceptual domains and, perhaps, of mathematics as a whole, may be seriously affected.

There is some intrinsic credibility in the idea of cascading failures in mathematical knowledge. Experience in the classroom leads one to recognize situations where the catastrophic collapse of a student's understanding does occur. A learner may seem to comprehend a mathematical topic well, and then something happens. Perhaps one too many idea is introduced, or some critical piece of background material is shaken, but suddenly the student's understanding of the concept falls apart. The

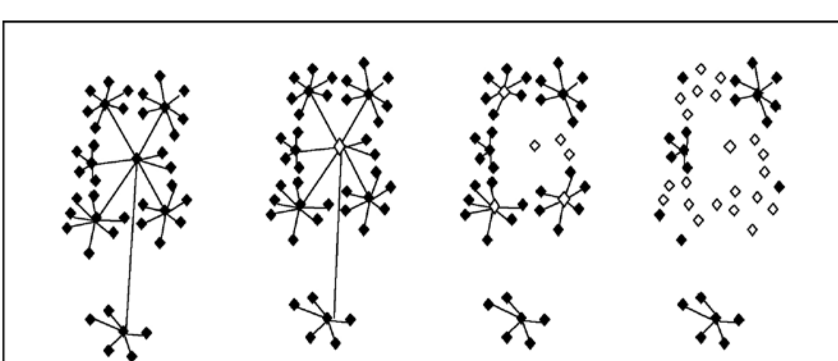

FIGURE 8. The cascading failure that occurs when a hub fails

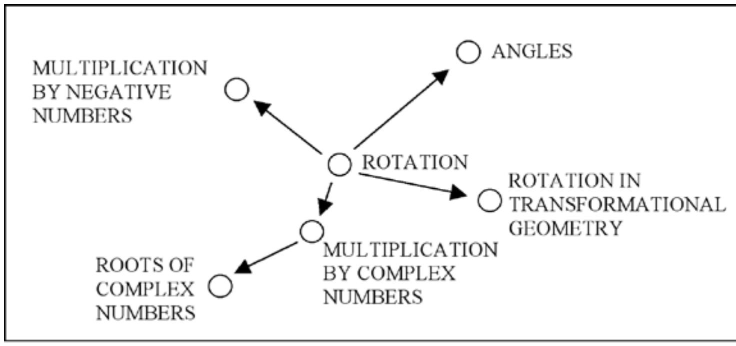

FIGURE 9. Conceptual domains linked to ROTATION

learner's comprehension of related concepts may be compromised as well. The student, whose knowledge of a basic domain—such as ROTATION—falters, will no longer fully understand aspects of connected ideas like ROTATIONS IN TRANSFORMATIONAL GEOMETRY or MULTIPLICATION BY COMPLEX NUMBERS (see Figure 9). Cascading failures are not just theoretical constructs, but do occur in real classrooms.

How might the robustness of the metaphoric network of mathematics be increased? Attention might be focused on strengthening a learner's grasp of key source domains. There are two difficulties with this approach. First, mathematicians, educators, and cognitive scientists do not understand a great deal about the network that represents mathematical knowledge. It is not yet known which conceptual domains are hubs in its structure. Second, while teachers could assist students to reinforce their conceptions of important source domains if these were known, the vulnerability that is characteristic of a scale-free network would not be eliminated. Hubs would still remain hubs.

In order to improve the robustness of a network, one must modify its structure. There are several ways in which this might be accomplished. Watts (2002) suggests that reducing the number of connections to important hubs would lessen the likelihood of network failure. When a hub does fail, fewer nodes will be affected, causing less disruption to the system as a whole. This approach is not one that a mathematics teacher could readily choose; particular concepts are repeatedly selected as source domains because of their usefulness and because the mathematics community has traditionally employed them to develop new ideas. It is not likely that a teacher would deliberately refuse to use domains that do provide a coherent structure for developing mathematical knowledge, nor would this be responsible.

It would seem that another approach is required. Barabási (2003) describes how increasing the number of connections among conceptual domains has the desired effect of reducing the network's dependence on its hubs. Adding even a few additional links—called *weak links* by network theorists—between clusters of nodes decreases the network's vulnerability (see Figure 10). The more distributed structure that results has sufficient redundancy to ensure that even if some nodes fail, alternative paths exist to maintain connections among the remaining nodes (Barabási, 2003; Buchanan, 2002).

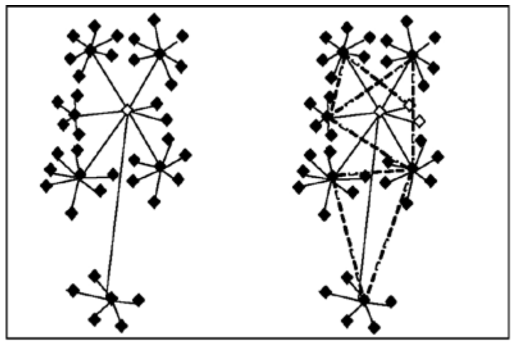

FIGURE 10. Decreased dependence on a central node as a result of adding a few weak links to a scale-free network

Implications for Mathematics Education

In the metaphoric network of mathematics, weak links could be established by increasing the number of metaphors used to make sense of a concept. If learners are not dependent on their comprehension of just one key source domain, then their understanding of mathematics should become more robust and not subject to the fragmentation and cascading failures that can impair scale-free networks. As Sfard (1997) asserts, "if our mathematical conceptions are to be sound and stable, they must stand on more than one metaphorical leg" (p. 367).

How can educators assist students to construct these metaphorical legs? First, more needs to be known about the cognitive mechanisms that constitute the network of mathematics. There are so many conceptual metaphors linking multitudes of conceptual domains that it is critical to know just what they are and how they are related. While some progress has been made (cf. Lakoff & Núñez, 2000), more research in this area is needed.

Second, educators should become aware of the many metaphors that connect mathematical concepts together. This is not easy; identification of metaphors requires sensitive attention to language, gestures, images, applications, and examples. Detecting the metaphors that underlie these representations would be difficult without suitable professional development.

Third, teachers could ensure that more than one metaphoric approach is used when introducing new concepts. Useful activities might highlight and reinforce the particular correspondences that exist between the source and target of a metaphor; as well, they might explore how these differ from one conceptual metaphor to another. Opportunities could also be provided for learners to practice shifting between metaphors as is necessitated by the changing context of problems.

Fourth, metaphors could become an inherent part of classroom discourse. Attention might be drawn to the use of metaphoric terms, notations and images, and students might be encouraged to discuss the associations these bring to mind. Learners might also be invited to articulate how various metaphors are different and similar. Such discussion might clarify the strengths and weaknesses of a metaphor when making sense of a particular concept and, thus, might assist learners to realize which metaphors are appropriate for use in different situations.

Finally, a reconceptualization of curriculum structures would seem desirable. All too often, programs of study focus on mathematical topics that are largely isolated from each other and arranged in an essentially linear fashion. Such a presentation inevitably conflicts with the notion of mathematics as a complex web of interconnected ideas. Programs of study could be built around the interdependence of mathematical concepts and highlight multiple metaphoric interpretations for each topic. Support for teachers could be provided with appropriate instructional materials and texts.

Conclusion

This paper discusses how network structures offer a fruitful model for interpreting mathematical knowledge. As the metaphoric network of mathematics possesses a scale-free topology, it is highly dependent on key concepts whose weakness can seriously compromise the robustness of the structure. The resulting failure of connected concepts, which spreads throughout the network, can have a deleterious effect on a student's comprehension of mathematics. Adding additional metaphoric links between conceptual domains—using multiple metaphors to give meaning to mathematical concepts—would prevent such cascading failures in understanding.

This network analysis of the structure of mathematical knowledge suggests that changes in both pedagogy and curriculum would be desirable. Concepts are important, but the connections between them are just as vital; it is the metaphoric links in mathematics that determine a concept's inferential structure, connect it to clusters of related ideas, and ensure its stability. As metaphors lead to new understandings, I would like to suggest that perceiving mathematics as the complex network of an evolving ecosystem, rather than as a tower of idea may provide a starting point for necessary and exciting changes in mathematics education.

Notes

1. Throughout the paper, I follow the convention of identifying conceptual domains, conceptual metaphors, and other cognitive mechanisms by using small capitals.
2. Metaphors will be identified using the convention 'TARGET' IS 'SOURCE'.
3. Figure 7 was my own attempt to illustrate the UNIT CIRCLE conceptual blend. Later, I discovered that it has a remarkable similarity to figures on pages 390–392 in Lakoff & Núñez's (2000) *Where Mathematics Comes From: How the embodied mind brings mathematics into being*. Independent development of the diagram illustrates how particular metaphors have entailments that compel certain interpretations. It is likely that any graphic representation of the UNIT CIRCLE conceptual blend would closely resemble Lakoff and Núñez's images.

References

Albert, R., Jeong, H., & Barabási, A. L. (2000). Attack and error tolerance of complex networks. *Review of Modern Physics 74*, 47–97.

Barabási, A. L. (2003). *Linked: How everything is connected to everything else and what it means for business, science, and everyday life.* New York: Plume.

Barabási, A. L. & Albert, R. (1999). Emergence of scaling in random networks. *Science, 286*, 509-512.

Barabási, A. L., Albert, R., Jeong, H. & Bianconi, G. (2000). Power-law distribution of the world wide web. *Science, 287*, 2115a.

Bianconi, G. & Barabási, A. L. (2001). Competition and multiscaling in evolving networks. *Europhysics Letters, 54*(4), 436-442.

Boyd, R. (1979). Metaphor and theory change: What is 'metaphor' a metaphor for? In A. Ortony (Ed.), *Metaphor and thought* (pp. 356-408). Cambridge: Cambridge University Press.

Buchanan, M. (2002). *Nexus: Small worlds and the groundbreaking theory of networks.* New York: Norton.

Chiu, M. M. (2000). Metaphorical reasoning: Origins, uses, development and interactions in mathematics. *Education Journal, 28*(1), 14-46.

Danziger, K. (1990). Generative metaphor and the history of psychological discourse. In D. E. Leary (Ed.), *Metaphors In The History Of Psychology* (pp. 331–356). Cambridge, UK: Cambridge University Press.

Davis, B. & Simmt, E. (2006). Mathematics-for-teaching: An ongoing investigation of the mathematics that teachers (need to) know. *Educational Studies in Mathematics, 61*(3), 293-319.

Davis, B. & Sumara, D. (2000). Curriculum forms: On the assumed shapes of knowing and knowledge. *Journal of Curriculum Studies, 32*(6), 821-845.

English, L. D. (1997). Analogies, metaphors, and images: Vehicles for mathematical reasoning. In L. D. English (Ed.), *Mathematical reasoning: Analogies, metaphors, and images* (pp. 3–18). Mahwah, NJ: Lawrence Erlbaum Associates.

Fauconnier, G., & Turner, M. (1998). Conceptual integration networks. *Cognitive Science, 22*(2), 133–187.

Gentner, D., Bowdle, B., Wolff, P. & Boronat, C. (2001). Metaphor is like analogy. In D. Gentner, K. J. Holyoak & B. N. Kokinov (Eds.), *The analogical mind: Perspectives from cognitive science* (pp. 199-253). Cambridge, MA: MIT Press.

Johnson, M. (1987). *The body in the mind: The bodily basis of meaning, imagination, and reason*. Chicago: University of Chicago Press.

Kimmel, M. (2002). *Metaphor, imagery, and culture: Spatialized ontologies, mental tools, and multimedia in the making*. Unpublished doctoral dissertation. Vienna: Department of Philosophy, University of Vienna.

Kövacses, Z. (2002). *Metaphor: A practical introduction*, Oxford, UK: Oxford University Press.

Lakoff, G. (1987). *Women, fire and dangerous things: What categories reveal about the mind*. Chicago: University of Chicago Press.

Lakoff, G. & Johnson, M. (1999). *Philosophy in the flesh: The embodied mind and its challenge to western thought*. New York: Basic Books.

Lakoff, G., & Núñez, R. (2000). *Where mathematics comes from: How the embodied mind brings mathematics into being*. New York: Basic Books.

Lamb, S. (1999). *Pathways of the brain: The neurocognitive basis of language*. Amsterdam: John Benjamin.

Meisner, M. (1992). Metaphors of nature: Old vinegar in new bottles? *Trumpeter, 9*(4). Retrieved March 11, 2005, from http://trumpeter.athabascau.ca/content/v9.4/meisner.html

Mowat, E. (2005). Embodied mathematics and education. *Delta-K, 42*(2), 20-29.

Mowat, E. & Davis, B. (in submission). *Interpreting Embodied Mathematics using Network Theory: Implications for Mathematics Education*.

Presmeg, N. C. (1997). Reasoning with metaphors and metonymies in mathematical learning. In L. D. English (Ed.), *Mathematical reasoning: Analogies, metaphors and images* (pp. 267-280). Mahwah, NJ: Lawrence Erlbaum.

Sfard, A. (1997). Commentary: On metaphorical roots of conceptual growth. In L. D. English (Ed.), *Mathematical reasoning: Analogies, metaphors, and images* (pp. 339–371). Mahwah, NJ: Lawrence Erlbaum Associates.

Watts, D. (2002). *Six degrees: The science of a connected age*. New York: W.W. Norton.

Watts, D., & Strogatz, S. (1998). Collective dynamics of 'small-world' networks. *Nature, 393*, 440–442.